Exercise 79

The **Bessel function** of order 0, $y = J(x)$, satisfies the differential equation $xy'' + y' + xy = 0$ for all values of x and its value at 0 is $J(0) = 1$.

(a) Find $J'(0) = 0$.

(b) Use implicit differentiation to find $J''(0)$.

Solution

Part (a)

Since $y = J(x)$ and $xy'' + y' + xy = 0$ holds for all values of x, set $x = 0$.

$$xJ''(x) + J'(x) + xJ(x) = 0$$

$$0[J''(0)] + J'(0) + 0[J(0)] = 0$$

$$J'(0) = 0$$

Part (b)

Differentiate both sides of the defining equation for $J(x)$ with respect to x.

$$\frac{d}{dx}(xy'' + y' + xy) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(xy'') + \frac{d}{dx}(y') + \frac{d}{dx}(xy) = 0$$

$$y'' + xy''' + y'' + y + xy' = 0$$

$$xy''' + 2y'' + xy' + y = 0$$

This holds for all values of x, so set $x = 0$ again.

$$xJ'''(x) + 2J''(x) + xJ'(x) + J(x) = 0$$

$$0[J'''(0)] + 2[J''(0)] + 0[J'(0)] + J(0) = 0$$

$$2J''(0) + J(0) = 0$$

Use the fact that $J(0) = 1$.

$$2J''(0) + 1 = 0$$

$$J''(0) = -\frac{1}{2}$$